Finite Volume Chiral Partition Functions and the Replica Method

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In the framework of chiral perturbation theory we demonstrate the equivalence of the supersymmetric and the replica methods in the symmetry breaking classes of Dyson indices $\beta = 1$ and $\beta = 4$. Schwinger-Dyson equations are used to derive a universal differential equation for the finite volume partition function in sectors of fixed topological charge, ν. All dependence on the symmetry breaking class enters through the Dyson index β. We utilize this differential equation to obtain Virasoro constraints in the small mass expansion for all β and in the large mass expansion for $\beta = 2$ with arbitrary ν. Using quenched chiral perturbation theory we calculate the first finite volume correction to the chiral condensate demonstrating how, for all βthere exists a region in which the two expansion schemes of quenched finite volume chiral perturbation theory overlap.

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I. INTRODUCTION

In the low-energy and chiral limit of QCD, chiral symmetry is spontaneously broken. The physics is dominated by the pseudo-Goldstone bosons, and the corresponding effective field theory is known as chiral perturbation theory. This theory is describable in terms of a chiral Lagrangian. However, the physics in this low-energy regime depends upon the exact way in which the chiral symmetry is spontaneously broken. There are believed to be three ways in which spontaneous chiral symmetry breaking can happen [1]. These depend upon the representation of the fermions in the following way:

- The representation of the fermions is pseudo-real. The expected symmetry breaking pattern is $SU(2N_f) \to Sp(2N_f)$.
- The representation is complex. In this case we expect the symmetry breaking pattern to be $SU_L(N_f) \times SU_R(N_f) \to SU(N_f)$. An example is fermions in the fundamental representation with the number of colours, $N_c \geq 3$. Thus ordinary QCD falls into this class.
- The fermions are in a real representation. The expected symmetry breaking pattern is then $SU(N_f) \to SO(N_f)$.

These symmetry breaking patterns are nowadays labeled by the Dyson-indices, $\beta = 1$, $\beta = 2$, and $\beta = 4$, respectively, due to a connection to Random Matrix theory [2]. For a symmetry breaking pattern $G \to H$, the fields live on the coset G/H. The usual perturbation scheme for chiral perturbation theory is for large volumes and is an expansion in terms of the momenta of the Goldstone modes. This perturbative scheme will be applied in most of this paper.

A systematic approach to calculations in (partially) quenched chiral perturbation theory is known as the supersymmetric method [3, 4, 5]. In the supersymmetric formulation k "valence" quark species are introduced with k ghost quarks of opposite (bosonic) statistics in addition to the N_f "sea" quarks. In this way the effective partition function is extended, becoming a generator of n-point functions with the additional quarks acting as source terms. The terminology "supersymmetric method" reflects the fact that the chiral flavor symmetry group is extended to a super Lie group, while space-time supersymmetry is not intrinsic to the method.

The replica method, which shall be applied in this paper, has turned out to be an alternative to the supersymmetric method. In this method one adds N_v fermionic valence quarks taking N_v to zero at the end of the calculations. Obviously, if we let $N_v \to 0$ we recover the original partition function. However, as in the supersymmetric method this extension of the partition function makes it a generating functional for n-point functions with the sources being the valence quarks. It can be convenient to add k sets of N_v valence quarks with masses $m_{v_1}, \ldots, m_{v_k}$, but for the applications in this paper we need only include one set of N_v valence quarks of mass m_v. We assume the symmetry breaking patterns in this extended theory to be the usual symmetry breaking patterns. Thus for intermediate calculations in this paper, in the above-mentioned symmetry breaking patterns we will simply replace N_f by $N_f + N_v$. Since the limit $N_v \to 0$ formally requires an analytic continuation of the Lie groups to non-integer dimensions the validity of the method is not obvious, but it can be shown that the required analytic continuation can be performed in series expansions [6, 7, 8]. Recently there have been some developments into non-perturbative results in this framework [9], but these are out of the scope of this paper. In both this framework and the supersymmetric one, the

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fully quenched limit corresponds to vanishing N_f while the theory is partially quenched for N_f non-vanishing.

The equivalence of these methods have been demonstrated for $\beta = 2$ and thus for ordinary QCD [6] by explicitly showing the equivalence of the Feynman rules at the one-loop level. In the first part of this paper we will extend this analysis to the other classes of chiral symmetry breaking, showing the equivalence only explicitly at the one-loop level but noting that the analysis is easily carried over to the higher loop level. To illustrate the equivalence we will also calculate the partially quenched chiral condensate at one-loop order, illustrating how the statistics signs of the supersymmetric formulation are reproduced in the replica formulation by the counting of possible quark-loops.

In the second part of this paper we present one of the main results of this paper, a universal second order differential equation to determine the finite volume chiral partition function in all classes of chiral symmetry breaking and for arbitrary topological charge, ν. This differential equation is written in terms of the fermion masses and all dependence on the pattern of symmetry breaking is through the Dyson index, β. The form of the differential equation in the specific case of $\beta = 2$ with zero topological charge is already known [10], but the generalization of this paper has not been found previously. The method is to develop Schwinger-Dyson equations for the effective low energy, or finite volume, partition function and next to realize how these can be written in terms of the fermion masses. For $\beta = 2$ with vanishing topological charge the effective partition function has been recognized [11] as a group integral of Kontsevich type with potential $\mathcal{V}(X) = 1/X$ and this model, also known as the one-link integral, has been intensively studied in the literature, see, e.g., refs. [8, 12, 13, 14].

Having found a governing differential equation we proceed to solve this equation. It should be realized that there are two relevant expansions, the small-mass and the large-mass expansions. Using the invariance properties of the effective finite volume partition function it is possible to find Virasoro constraints. For the small mass phase this was first done in ref. [11] in the simplest version, namely $\beta = 2$ with vanishing topological charge, by recognizing the partition function as being a unitary integral of Kontsevich type. More recently, Virasoro constraints in the small mass phase for general β and topological charge have been calculated in ref. [7] by using a method much similar to the one presented here. We exactly reproduce the Virasoro constraints of ref. [7]. The large mass expansion is more involved as it is not possible to do a simple perturbative expansion, instead the large mass expansion is an asymptotic, or saddle-point, expansion [10, 11]. The saddle-point approximation corresponds to the classical limit and is an expansion in powers of $1/N_f$. In the large mass limit Virasoro constraints have been previously determined in the case of $\beta = 2$ [10]. We find Virasoro constraints in the general case of $\beta = 2$, where the inclusion of a non-vanishing topological charge is thus a new result. At next to leading order in $1/N_f$ the saddle-point approximation turns out to be a simple correction to the leading order term in addition to an (apparently) infinite expansion in the inverse fermion masses. However, this last expansion is proportional to $1 - 2/\beta$ and thus it vanishes if $\beta = 2$. Precisely this property allows us to calculate large-mass Virasoro constraints in the $\beta = 2$ case with arbitrary topological charge. Although we do not determine Virasoro constraints in the other classes of symmetry breaking we find that it is nevertheless possible to extract useful information from the saddle-point approximation; in the case of equal fermion masses the differential equation for the partition function is still tractable. We solve the governing differential equation for equal fermion masses and as an application we find the lowest order corrections to the fully quenched chiral condensate in sectors of topological charge, a result which will be very useful in the last part of this paper.

As Gasser and Leutwyler [15] were the first to point out, there are two finite volume regimes to consider. The first is the large, but finite, volume which can be considered a small perturbation to the infinite volume theory. This is the volume considered in the first parts of this paper and to which the usual chiral perturbation theory in terms of a momentum expansion of the chiral Lagrangian applies. The last part of the paper concerns quenched chiral perturbation theory in volumes much smaller than the correlation lengths of the Goldstone modes. Although this may seem ill-defined at first it is still possible by means of numerical simulations to extract physical, infinite volume quantities from this theory. In this volume-regime the usual momentum expansion breaks down due to the presence of zero-momentum modes [16]. The different expansion scheme required in this regime, known as the ϵ-expansion [15], treats this problem by means of a collective field technique, collecting the zero-momentum modes in one field and the non-zero modes in another. The non-zero modes are still perturbative. Difficulties in this scheme arise from the fact that the integration over the zero-momentum modes has to be done exactly.

In ref. [17] it was shown that in the case of $\beta = 2$ it is possible to determine a region in which the two perturbative expansion schemes overlap also in the quenched limit. We extend this analysis to the other classes of chiral symmetry breaking, seeing how also in these cases such a region exists and demonstrating how the quenched chiral condensates of the two regimes exactly match in the above-mentioned region.

This paper is organized as follows. In Section II we consider (partially) quenched chiral perturbation theory. The equivalence of the supersymmetric and the replica method in the alternative ($\beta = 1, 4$) classes of spontaneous chiral symmetry breaking is explicitly demonstrated at the one-loop level. Since equivalence was demonstrated for ordinary QCD ($\beta = 2$) in ref. [6], these two approaches to calculations in (partially) quenched chiral perturbation theory are equivalent in all classes of spontaneous chiral symmetry breaking.

Section III concerns finite volume chiral perturbation theory. Schwinger-Dyson equations are derived for the finite volume chiral partition functions and utilized to obtain Virasoro constraints in the limit of small quark masses and also (for $\beta = 2$) in the limit of large quark masses. For general β we find the next to leading order term in a mass expansion of the partition function. Although the calculations of this section are not considered to be in a quenched setting, the relevance of these results in such a setting is demonstrated by calculating the first correction in a mass expansion to the quenched chiral condensate.

We return to considering quenched chiral perturbation theory in section IV. Here it is demonstrated how a volume-regime exists in which the two perturbation schemes of finite volume chiral perturbation theory match in the quenched approximation.

II. THE EQUIVALENCE OF THE SUPERSYMMETRIC AND THE REPLICA METHODS

It has been shown that an alternative approach towards calculations in partially quenched chiral perturbation theory is offered by the replica method. The equivalence between the supersymmetric method and the replica method has been demonstrated for $\beta = 2$ in ref. [6]. In chiral perturbation theory the natural way of demonstrating the equivalence between the methods is demonstrating how the generating functional in the replica formulation results in Feynman rules equivalent to those obtained from the generating functional in the supersymmetric formulation. In this section we will extend this equivalence to the other classes of chiral symmetry breaking. As was the case in ref. [6] we will only demonstrate this equivalence at the one-loop level and thus the $\mathcal{O}(p^2)$ expansion of the Lagrangian density is sufficient with the $\mathcal{O}(p^4)$ terms acting as counter terms. To demonstrate the equivalence, we compare our results with those obtained in ref. [18] by means of the supersymmetric method. The extension to the $\mathcal{O}(p^4)$ Lagrangian is not very difficult, as it only affects the vertices, and not the propagator structure.

In the framework of chiral perturbation theory, the effective partition function is given by

$$\mathcal{Z} = \int_{U \in G/H} dU \; e^{-\int d^4x \mathcal{L}_{\rm eff}} \tag{1}$$

with the usual effective chiral Lagrangian density to second order in momenta

$$\begin{aligned} \mathcal{L}_{\rm eff} \;=\; & \frac{F_\beta^2}{4}{\rm Tr}(\partial_\mu U^\dagger \partial_\mu U) - \frac{\Sigma_0}{2}{\rm Tr}(\mathcal{M}^{(\beta)}(U + U^\dagger)) \\ & + \frac{m_0^2}{2N_c}\Phi_0^2 + \frac{\alpha}{2N_c}\partial_\mu \Phi_0 \partial_\mu \Phi_0. \end{aligned} \tag{2}$$

The Goldstone fields U are parameterized as

$$U = \exp(i\sqrt{2}\Phi/F_\beta). \tag{3}$$

The last two terms in (2) contain the flavor singlet field $\Phi_0 \equiv {\rm Tr}\Phi$. In the supersymmetric formulation this term is an invariant of the reduced symmetry group and it has to be included in the replica Lagrangian as well, since otherwise the quenched replica limit would not exist [27]. For the time being we will ignore the term proportional to α and reinstate it at the end of this section by substituting m_0^2 with $m_0^2 + \alpha p^2$.

The pion decay constant F_β is chosen so that the fields satisfy the usual Gell-Mann–Oakes–Renner relation for the masses of the mesons

$$M_{ij}^2 \;=\; (m_i + m_j)\frac{\Sigma_0}{F^2}. \tag{4}$$

With our choice of normalization of the fields it turns out that

$$F_2 = F_4 = F \tag{5}$$

$$F_1 = F/\sqrt{2}. \tag{6}$$

Notice that the fact that our choices of normalization of the fields and of F_β differ from those of ref. [18] does not stem from any peculiarities of the replica method. Rather, we find these choices more natural, since they ensure that all complex fields are normalized in the same way.

The infinite volume chiral condensate is denoted by Σ_0 and in the replica formulation the partially quenched chiral condensate of the additional fermionic copies is, to one-loop order, given by

$$\begin{aligned} \Sigma(m_v, \{m\}) \;&=\; \frac{1}{V}\lim_{N_v \to 0}\frac{1}{N_v}\frac{\partial}{\partial m_v}\ln \mathcal{Z} \\ &=\; \lim_{N_v \to 0}\frac{1}{N_v}\left\langle \partial_{m_v}(-\mathcal{L})\right\rangle. \end{aligned} \tag{7}$$

A. $\beta = 1$

The expected symmetry breaking of fermions in a pseudo-real representation is $SU(2N_f + 2N_v) \to Sp(2N_f + 2N_v)$ with Dyson index $\beta = 1$.

Following ref. [1] we choose an explicit representation of the field Φ as

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi & \psi \\ \psi^\dagger & \phi^T \end{pmatrix}. \tag{8}$$

The normalization is taken for convenience. In this representation the field ϕ is Hermitian while the field ψ is complex and anti-symmetric. Both fields are written in terms of $(N_f + N_v) \times (N_f + N_v)$ matrices.

The mass matrix can be chosen diagonal with

$$\begin{aligned} \mathcal{M}^{(1)} \;=\; \tfrac{1}{2}\,{\rm diag}(\;& m_1, \ldots, m_{N_f}, \overbrace{m_v, \ldots, m_v}^{N_v}, \\ & m_1, \ldots, m_{N_f}, \overbrace{m_v, \ldots, m_v}^{N_v}\;), \end{aligned} \tag{9}$$

where this normalization ensures that the Gell-Mann–Oakes–Renner relation (4) is satisfied.

To calculate the chiral condensate to one-loop order we evaluate the Lagrangian to second order in the fields. Only the term containing m_0^2 mixes states and this only applies to the diagonal fields. This makes it easy to read off the propagators for the "off-diagonal" mesons. These are seen to be

$$D_{ij} = \frac{1}{p^2 + M_{ij}^2}. \quad (10)$$

The diagonal mesons also inherit this part, so that their propagator becomes

$$G_{ij}^{-1} = \delta_{ij}(p^2 + M_{ii}^2) + 2m_0^2/N_c. \quad (11)$$

This expression can be inverted by noting that for general matrices A, B, with $A_{ij} = a$ constant, and $B_{ij} = \delta_{ij} b_{ii}$ (no sum over i) diagonal,

$$(A+B)^{-1} = B^{-1} - B^{-1}AB^{-1}\frac{1}{1 + a\sum_{i=1}^{\mathcal{N}} b_{ii}^{-1}}. \quad (12)$$

Thus we find that

$$G_{ij} = \frac{\delta_{ij}}{p^2 + M_{ij}^2} - \frac{2m_0^2/N_c}{(p^2 + M_{ii}^2)(p^2 + M_{jj}^2)\mathcal{F}^{\beta=1}(p^2)}, \quad (13)$$

where

$$\begin{aligned}\mathcal{F}^{\beta=1} &= 1 + \frac{2m_0^2}{N_c}\sum_{k=1}^{N_f+N_v}(p^2 + M_{kk}^2)^{-1} \\ &= 1 + \frac{2m_0^2}{N_c}\left(\frac{N_v}{p^2 + M_{vv}^2} + \sum_{k=1}^{N_f}\frac{1}{p^2 + M_{kk}^2}\right), \quad (14)\end{aligned}$$

since the masses of the mesons in the replica set are identically m_v. These expressions for the propagators can be compared with the corresponding expressions from the supersymmetric method obtained in [18]. For $N_v \to 0$ the expressions are equivalent, the difference arising from our alternative normalization conventions, only.

We now proceed to calculate the chiral condensate to one-loop order. It is given by (7) and can thus be calculated directly from the second order expansion of the Lagrangian. We find that

$$\begin{aligned}\frac{\Sigma(m_v, \{m\})}{\Sigma_0} &= \frac{1}{V\Sigma_0}\lim_{N_v\to 0}\frac{1}{N_v}\frac{\partial}{\partial m_v}\ln \mathcal{Z} \\ &= \lim_{N_v\to 0}\frac{1}{N_v}\left[N_v - \frac{2}{F^2}\left(N_v\sum_{f=1}^{N_f}\Delta_{vs}\right.\right. \\ &\quad \left.\left. + N_v(N_v-1)\Delta_{vv} + \frac{N_v}{2V}\sum_p G_{vv}(p^2)\right)\right] \\ &= 1 - \frac{1}{F^2}\left(2N_f\Delta_{vs} - 2\Delta_{vv} + \frac{1}{V}\sum_p G_{vv}\right), \quad (15)\end{aligned}$$

where the pion propagator is given by

$$\Delta_{ij} \equiv \frac{1}{V}\sum_p \frac{1}{p^2 + M_{ij}^2} = \frac{1}{V}\sum_p D_{ij}(p^2), \quad (16)$$

and where we in the last step for simplicity have put the sea quark masses equal. This result is identical to the result obtained from the supersymmetric formulation [18].

B. $\beta = 4$

For fermions in a real representation the expected symmetry breaking class is $SU(N_f + N_v) \to SO(N_f + N_v)$ corresponding to the Dyson index $\beta = 4$. In this case an explicit representation of the Goldstone bosons is [18]

$$\Phi = \begin{pmatrix} A_{11} & \frac{1}{\sqrt{2}}A_{12} & \cdots & \frac{1}{\sqrt{2}}A_{1,N_f+N_v} \\ \frac{1}{\sqrt{2}}A_{12} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{1}{\sqrt{2}}A_{1,N_f+N_v} & \cdots & \cdots & A_{N_f+N_v,N_f+N_v} \end{pmatrix}. \quad (17)$$

Φ is a real, symmetric matrix [1]. Thus the normalization of the off-diagonal mesons differs from that of the diagonal mesons to ensure correctly normalized kinetic terms in the Lagrangian. The mass matrix is diagonal with

$$\mathcal{M}^{(4)} = \mathrm{diag}(m_1, \ldots, m_{N_f}, \overbrace{m_v, \ldots, m_v}^{N_v}). \quad (18)$$

Following the steps above we evaluate the Lagrangian (2) to second order in the fields to obtain the one-loop correction to the chiral condensate. As for $\beta = 1$ the propagators of the off-diagonal mesons are easy to read off. Again we find

$$D_{ij} = \frac{1}{p^2 + M_{ij}^2}. \quad (19)$$

The propagators of the diagonal mesons are

$$G_{ij}^{-1} = \delta_{ij}(p^2 + M_{ii}^2) + m_0^2/N_c \quad (20)$$

which, inverted by the use of eq. (12), yields

$$G_{ij} = \frac{\delta_{ij}}{p^2 + M_{ij}^2} - \frac{m_0^2/N_c}{(p^2 + M_{ii}^2)(p^2 + M_{jj}^2)\mathcal{F}^{\beta=4}(p^2)}, \quad (21)$$

with

$$\mathcal{F}^{\beta=4} = 1 + \frac{m_0^2}{N_c}\left(\frac{N_v}{p^2 + M_{vv}^2} + \sum_{k=1}^{N_f}\frac{1}{p^2 + M_{kk}^2}\right). \quad (22)$$

In the limit $N_v \to 0$ this result also agrees with the result obtained in ref. [18] from the supersymmetric formulation.

The calculation of the partially quenched chiral condensate proceeds as before. From (7) we find

$$\frac{\Sigma(m_v,\{m\})}{\Sigma_0} = 1-\frac{1}{F^2}\left(\frac{1}{2}N_f\Delta_{vs}-\frac{1}{2}\Delta_{vv} +\frac{1}{V}\sum_p G_{vv}(p^2)\right), \quad (23)$$

with the pion propagators given as in eq. (16) and the sea quark masses equal. Again, this result exactly matches the result from the supersymmetric method [18].

C. The chiral condensate in all classes of chiral symmetry breaking

Including the chiral condensate in theories with fermions in a complex representation ($\beta = 2$) [19, 20], allows us to write, for general β,

$$\frac{\Sigma^\beta(m_v,\{m\})}{\Sigma_0} = 1-\frac{1}{F^2}\left(\frac{2}{\beta}N_f\Delta_{vs}-\frac{2}{\beta}\Delta_{vv} +\frac{1}{V}\sum_p G_{vv}(p^2)\right), \quad (24)$$

with the propagator for the neutral mesons

$$G_{ij} = \frac{\delta_{ij}}{p^2+M_{ij}^2}-\frac{(1+\delta_{\beta,1})(m_0^2+\alpha p^2)/N_c}{(p^2+M_{ii}^2)(p^2+M_{jj}^2)\mathcal{F}^\beta(p^2)}. \quad (25)$$

For general β

$$\mathcal{F}^\beta = 1+(1+\delta_{\beta,1})\frac{m_0^2+\alpha p^2}{N_c}\left(\frac{N_v}{p^2+M_{vv}^2} +\sum_{k=1}^{N_f}\frac{1}{p^2+M_{kk}^2}\right). \quad (26)$$

Here the α dependence has been reinserted. Note that the first term in $\frac{1}{V}\sum_p G_{vv}$ is simply Δ_{vv} which in the case of $\beta = 2$ exactly cancels the propagator Δ_{vv} in (24). Cancellations like this is a typical property of $\beta = 2$ and we shall see several additional examples of terms proportional to the factor $\delta_\beta \equiv 1 - 2/\beta$, terms which only cancel in the case of $\beta = 2$. Explicitly

$$\delta_\beta \equiv 1-\frac{2}{\beta} = \begin{cases} -1, & \beta=1 \\ 0, & \beta=2 \\ \frac{1}{2}, & \beta=4 \end{cases} \quad (27)$$

Taking advantage of the above we find that we can write the chiral condensate for general β as

$$\frac{\Sigma^\beta(m_v,\{m\})}{\Sigma_0} = 1-\frac{1}{F^2}\left(\frac{2}{\beta}N_f\Delta_{vs}+\delta_\beta\Delta_{vv} -\frac{1}{V}\sum_p \frac{(1+\delta_{\beta,1})(m_0^2+\alpha p^2)/N_c}{(p^2+M_{vv}^2)(p^2+M_{vv}^2)\mathcal{F}^\beta(p^2)}\right). \quad (28)$$

III. SCHWINGER-DYSON EQUATIONS AND VIRASORO CONSTRAINTS

Having established the equivalence of the replica and the supersymmetric methods in all classes of chiral symmetry breaking we now turn towards calculating Schwinger-Dyson equations that govern the behavior of the partition functions in the three classes of chiral symmetry breaking. That is, we turn from (partially) quenched chiral perturbation theory to ordinary chiral perturbation theory, but the results are also useful for the quenched approximation. It turns out that the Schwinger-Dyson equations can be expressed in a universal way.

The method is here shortly outlined. For greater details see Appendix A. When working with ordinary QCD ($\beta = 2$) the finite volume partition function is an integral over the unitary group [15, 21]

$$\mathcal{Z}_\nu^{\beta=2} = \int_{U\in U(N_f)} dU(\det U)^\nu e^{\frac{1}{2}\mathrm{Tr}(U\mathcal{M}^\dagger+U^\dagger\mathcal{M})}. \quad (29)$$

Here we have included the volume and the chiral condensate in $\mathcal{M} = MV\Sigma_0$, where M is the usual quark mass matrix.

The partition function is a periodic function in the vacuum angle, θ. This means that the full partition function $\mathcal{Z}(\theta)$ may be expressed as a Fourier series, where the expansion coefficients, $\mathcal{Z}_\nu$, correspond to the partition function in sectors of fixed topological charge, ν.

As discussed by Leutwyler and Smilga [21], the topological charge need not be integer. Letting n_L and n_R denote the number of left- and right-handed zero-modes of the covariant derivative $\not{D}$, respectively, the index theorem states that

$$n_L-n_R = \frac{1}{16\pi^2}\int d^4x\,\mathrm{Tr}(G_{\mu\nu}\tilde{G}_{\mu\nu}). \quad (30)$$

The trace depends upon the fermionic representation, and the trace of the generators, t^a, is $\frac{1}{2}\ell(r)\delta^{ab} = \mathrm{Tr}[t^at^b]$. $\ell(r)$ is the index of the fermionic representation.

The topological charge is

$$\nu = \frac{1}{32\pi^2}\int d^4x(G^a_{\mu\nu}\tilde{G}^a_{\mu\nu}), \quad (31)$$

and thus it is possible to deduce the relation

$$\nu = \bar{\nu}\frac{1+\delta_{\beta,4}}{\ell(r)}, \quad (32)$$

where $\bar{\nu}$ is integer. The reason for the appearance of $\delta_{\beta,4}$ in eq. (32) is that for real representations, the reality of the Dirac operator leads to all the eigenfunctions of γ_5 appearing in pairs. In particular, the number of zero-modes is even for $\beta = 4$.

For the gauge groups and representations usually considered in the literature the relation between ν and $\bar{\nu}$ is simple. For $\beta = 1$ one usually considers fermions in the fundamental representation of $SU(N_c = 2)$ for which $\ell(r) = 1$ and $\nu = \bar{\nu}$. The same is the case for ordinary QCD, the fundamental representation of $SU(N_c = 3)$. However, for $\beta = 4$ the usual scenario is adjoint (Majorana) fermions for which $\ell(r) = 2N_c$ and $\bar{\nu} = N_c\nu$ (again with the gauge group $SU(N_c)$). For simplicity, these are also the only representations considered here, but it is straightforward to generalize to other representations.

In the following, the crucial property of the partition function, eq. (29), is that $(\det \mathcal{M})^{-\nu} \mathcal{Z}_\nu^{\beta=2}$ only depends on the N_f eigenvalues x_i, $i = 1, \ldots, N_f$, of $\mathcal{M}\mathcal{M}^\dagger$ [7]. This can be seen from rescaling $\mathcal{M} \to V^{-1}\mathcal{M}$ and using the invariance of the unitary measure under left and right multiplication.

This dependence on the eigenvalues of $\mathcal{M}\mathcal{M}^\dagger$ only also holds in the other classes of chiral symmetry breaking [7]. In the case of $\beta = 1$ the symmetry breaking pattern is $SU(2N_f) \to Sp(2N_f)$ and a convenient parametrization of the Goldstone fields is UIU^t with the partition function [7]

$$\mathcal{Z}_\nu^{\beta=1} = \int_{U \in U(2N_f)} dU (\det U)^\nu e^{\frac{1}{4}\mathrm{Tr}((UIU^t)^\dagger \tilde{\mathcal{M}} + UIU^t \tilde{\mathcal{M}}^\dagger)}. \quad (33)$$

I is the anti-symmetric unit matrix and $\tilde{\mathcal{M}}$ is an arbitrary anti-symmetric complex matrix. Concentrating on theories without di-quark terms, we have

$$I = \begin{bmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{bmatrix}, \quad \tilde{\mathcal{M}} = \begin{bmatrix} 0 & \mathcal{M} \\ -\mathcal{M}^t & 0 \end{bmatrix}, \quad (34)$$

where we again define $\mathcal{M} = MV\Sigma_0$ with M being the usual $N_f \times N_f$ quark mass matrix. In this case $(\det \mathcal{M})^{-\nu} \mathcal{Z}_\nu^{\beta=1}$ is a symmetric function of the eigenvalues of $\tilde{\mathcal{M}}\tilde{\mathcal{M}}^\dagger$ [7].

In the case of $\beta = 4$ the Goldstone fields can be parametrized by UU^t, $U \in U(N_f)$. With $\bar{\nu} \equiv N_c\nu$ the finite volume partition function is [21, 22]

$$\mathcal{Z}_\nu^{\beta=4} = \int_{U \in U(N_f)} dU (\det U)^{2\bar{\nu}} e^{\frac{1}{2}\mathrm{Tr}((UU^t)^\dagger \mathcal{M} + UU^t \mathcal{M}^\dagger)}, \quad (35)$$

again with $\mathcal{M} = MV\Sigma_0$ but this time with $\mathcal{M}$ being a symmetric matrix. $(\det \mathcal{M})^{-\bar{\nu}} \mathcal{Z}_\nu^{\beta=4}$ is a symmetric function of $\mathcal{M}\mathcal{M}^\dagger$ [7].

Useful Schwinger-Dyson equations are obtained in Appendix A. We multiply the integrands in the partition functions by $\mathrm{Tr} t^a U \mathcal{M}^\dagger$ ($\beta = 2$), $\mathrm{Tr} t^a UIU^t \mathcal{M}^\dagger$ ($\beta = 1$) and $\mathrm{Tr} t^a UU^t \mathcal{M}^\dagger$ ($\beta = 4$), and then subsequently act with the differential operator ∇^a. The generators of the integration manifold t^a are normalized according to $t^a_{ij} t^a_{kl} = \frac{1}{2}\delta_{il}\delta_{jk}$. Since the integral of a total derivative vanishes this gives relations among the expectation values resulting in the Schwinger-Dyson equations (A42), (A44), and (A46) given in the appendix.

These equations can be written in a remarkably uniform way. Note that the number of generators, $\mathcal{N}$, of the cosets is 1 higher than the dimension of the "original" cosets stated in the introduction because of the projection onto sectors of fixed topological charge. For instance for $\beta = 2$ the integration goes from an integration over $SU(N_f)$ to an integration over $U(N_f)$ by including the topological charge. The dimensions of the "original" cosets are stated in Table I. Gathering the differential equations deduced in Appendix A we see that it is possible to write the differential equations universally as

$$\left[x_i^2 \partial_i^2 + \left(\frac{\mathcal{N}}{N_f} + \nu \right) x_i \partial_i + \frac{2}{\beta} \sum_{\substack{i,j \\ i \neq j}} \frac{x_i x_j}{x_i - x_j} (\partial_i - \partial_j) \right] \left(\prod_k x_k^{-\nu/2} \right) \mathcal{Z}_\nu^\beta = \frac{1}{4} \sum_i x_i \left(\prod_k x_k^{-\nu/2} \right) \mathcal{Z}_\nu^\beta, \quad (36)$$

with $\nu \to \bar{\nu}$ for $\beta = 4$. This differential equation is one of the main results of the present paper, nicely combining the seemingly very different integrations over cosets into one governing differential equation in terms of the fermion masses. All coset dependence is thus included in the single parameter β. In writing this differential equation we have assumed that the mass matrix $\mathcal{M}$ is real with positive eigenvalues allowing us to write $(\det \mathcal{M})^{-\nu} = \prod_i x_i^{-\nu/2}$.

In the following it will be convenient to introduce the notation

$$\frac{\mathcal{N}}{N_f} = \frac{2}{\beta} N_f + \delta_\beta. \quad (37)$$

An alternative way of obtaining useful Schwinger-Dyson equations, reminiscent of the one being presented here, is to insert a δ_{ik} in the partition function in the form of $U_{ij} U^\dagger_{jk}$, $U \in U(N_f)$, subsequently realizing that this can be reexpressed as a differential equation, the differentiation being with respect to the mass matrices. However, one should realize that when recognizing $\langle UU^\dagger \rangle$ as a differential operator, in the cases of $\beta = 1, 4$ the fact that the mass matrices are anti-symmetric and symmetric, respectively, causes complications with regard to the diagonal elements. This is especially so for the case of $\beta = 4$, see for instance eq. (A32). This method of obtaining Schwinger-Dyson equations has for example been employed in ref. [7] to obtain the Virasoro constraints in the small mass phase. Thus, in eqs. (19) and (28) of ref. [7] a factor of 2 is implied for each differentiation with respect to a diagonal element. Having deduced the differential equation (36), in which all dependence upon the mass matrices is through the eigenvalues, we need no longer be concerned with this point.

β	Coset	$\mathcal{N}-1$
1	$SU(2N_f)/Sp(2N_f)$	$N_f(2N_f-1)-1$
2	$SU_L(N_f)\times SU_R(N_f)/SU(N_f)$	N_f^2-1
4	$SU(N_f)/SO(N_f)$	$N_f\frac{N_f+1}{2}-1$

TABLE I: The number of generators of the cosets in the three classes of chiral symmetry breaking.

A. The small mass phase

We now proceed to find Virasoro constraints for the finite volume partition function by expanding in the eigenvalues of $\mathcal{M}\mathcal{M}^\dagger$. For the small mass phase this was first done in ref. [11] in the simplest version, namely $\beta = 2$ with vanishing topological charge. More recently, Virasoro constraints in the small mass phase have been calculated in ref. [7]. We present our calculation of the small mass Virasoro constraints here mainly for two reasons. We wish to demonstrate the ease by which we arrive at the Virasoro constraints by using eq. (36) and to demonstrate the correctness of (36) by exactly reproducing the results of [7].

In the small mass phase a suitable set of expansion parameters for the partition function is defined by

$$t_k \equiv \frac{1}{4^k k}\sum_i x_i^k, \tag{38}$$

which transforms the derivatives in (36) into

$$\frac{\partial}{\partial x_i} = \sum_k \frac{1}{4^k} x_i^{k-1}\frac{\partial}{\partial t_k}, \tag{39}$$

$$\frac{\partial^2}{\partial x_i^2} = \sum_k \frac{1}{4^k}(k-1)x_i^{k-2}\frac{\partial}{\partial t_k} + \sum_{k,l}\frac{1}{4^{k+l}}x_i^{k+l-2}\frac{\partial}{\partial t_k}\frac{\partial}{\partial t_l}. \tag{40}$$

Note that using the invariance of the Haar measure as well as permutation symmetry of the x_i, we are allowed to remove the sum over i [12]. Inserting the derivatives in our master differential equation (36) and removing the sum over i, we obtain

$$\Big[\textstyle\sum_k \frac{1}{4^k}(k-1)x_i^k\frac{\partial}{\partial t_k} + \sum_{k,l}\frac{1}{4^{k+l}}x_i^{k+l}\frac{\partial}{\partial t_k}\frac{\partial}{\partial t_l} + \left(\frac{2}{\beta}N_f+\delta_\beta+\nu\right)\sum_k\frac{1}{4^k}x_i^k\frac{\partial}{\partial t_k} + \frac{2}{\beta}\sum_{k=2}^{\infty}\sum_{l=1}^{k-1}(k-l)\frac{1}{4^l}x_i^l t_{k-l}\frac{\partial}{\partial t_k} - \frac{2}{\beta}\sum_{k=2}^{\infty}(k-1)\frac{1}{4^k}x_i^k\frac{\partial}{\partial t_k} - \frac{1}{4}x_i\Big] \times \left(\prod_{i=1}^{N_f}x_i\right)^{-\nu/2}\mathcal{Z}_\nu^\beta = 0. \tag{41}$$

Determining the Virasoro constraints is now a matter of reading off the coefficients of x_i^k. This procedure is justified provided the coefficients are determined uniquely and in a consistent way. Uniqueness is satisfied by requiring the boundary condition $\mathcal{Z}_\nu^\beta = 1$ if all fermion masses vanish. Consistency of the equations obtained by simply reading off the coeffecients of x_i^k is a more subtle point, all of the expansion coefficients t_k being independent only in the $N_f \to \infty$ limit. That this procedure is consistent has been demonstrated in ref. [11]. We thus find the constraints

$$\left[L_n^\beta - \frac{\beta}{2}\delta_{1,n}\right]\left(\prod_i x_i\right)^{-\nu/2}\mathcal{Z}_\nu^\beta = 0, \tag{42}$$

with the Virasoro operators L_n^β

$$L_n^\beta = \left(N_f + \frac{\beta}{2}(n\delta_\beta+\nu)\right)\frac{\partial}{\partial t_n} + \frac{\beta}{2}\sum_{m=1}^{n-1}\frac{\partial}{\partial t_m}\frac{\partial}{\partial t_{n-m}} + \sum_{m=1}^{\infty} m t_m\frac{\partial}{\partial t_{m+n}}. \tag{43}$$

Again we have $\nu \to \bar\nu$ for $\beta = 4$. The normalization is chosen such that the operators L_n^β satisfy the Virasoro algebra without central charge

$$[L_m^\beta, L_n^\beta] = (m-n)L_{m+n}^\beta, \qquad m,n \geq 1. \tag{44}$$

The virtue of the Virasoro operator formalism is that because the operators satisfy the Virasoro algebra, in principle all coefficients of a Taylor expansion in terms of the parameters (38) can be found from L_1 and L_2. The Virasoro operators (43) exactly match those found in ref. [7]. For the solution to the constraints we also refer the reader to ref. [7].

B. The saddle-point approach to the large mass phase

Virasoro constraints in the large mass phase for $\beta = 2$ with vanishing topological charge were found in ref. [10]

and solved in ref. [8]. Below we will present the extension of these results to include a non-vanishing topological charge. We will also give the first terms in a mass expansion of the $\beta = 1, 4$ partition functions in the case of equal quark masses. Analytical results with equal quark masses have been found in both of these symmetry breaking classes [22]. The disadvantage of these results is that they depend on determinants proportional in size to N_f which quickly makes calculations somewhat cumbersome. The results presented here, on the other hand, have a very simple N_f dependence.

The difficulty in the large-mass expansion arises from the fact that it is not possible to express the partition function in terms of a simple Taylor expansion in the inverse masses. However, as noted in the literature [10, 11], for $\beta = 2$ this becomes possible after extracting a prefactor from the partition function. In this case the prefactor turns out to be given by the saddle-point approximation, or the classical contribution, of the partition function and thus one would expect this to be the case in general. The classical contribution is defined as the $N_f \to \infty$ limit. We have been able to determine the classical contribution to the next to leading order only in the case of $\beta = 2$. For $\beta = 2$ this is the order needed to be able to include the topological charge in the Virasoro constraints. In the case of $\beta = 1, 4$ however, we have found indications that the next to leading order term in the saddle-point approximation is itself given by an infinite mass-expansion. Even so, using the lowest order correction in this mass-expansion we have calculated the lowest order mass dependence of the partition function for $\beta = 1, 4$ with equal fermion masses and non-vanishing topological charge. Using this result we calculate the one-loop correction to the quenched chiral condensate.

Following [10, 14] we define the free energy, F, in sectors of vanishing topological charge by $\mathcal{Z}_0^\beta = e^{N_f F}$. Defining in general $\mathcal{Z}_\nu^\beta = e^{N_f F_\nu}$ we find it useful to introduce

$$G_\nu \;=\; F_\nu - \frac{1}{N_f}\frac{\nu}{2}\sum_i \ln x_i, \tag{45}$$

since then $(\prod_i x_i^{-\nu/2})\mathcal{Z}_\nu^\beta = e^{N_f G_\nu}$. For the differential operators this means

$$\frac{\partial}{\partial x_i} e^{N_f G_\nu} \;=\; N_f e^{N_f G_\nu}\frac{\partial}{\partial x_i} G_\nu \tag{46}$$

$$\frac{\partial^2}{\partial x_i^2} e^{N_f G_\nu} \;=\; N_f e^{N_f G_\nu}\left[N_f\left(\frac{\partial G_\nu}{\partial x_i}\right)^2 + \frac{\partial^2 G_\nu}{\partial x_i^2}\right]. \tag{47}$$

In what follows we will use the somewhat sloppy terminology that G_ν is the free energy.

The free energy, being an evaluation around the $N_f \to \infty$ saddle-point, can be written as a power expansion in $1/N_f$. When doing the counting of orders of N_f the action counts as order N_f. Explicitly including this factor to ease the counting of powers, when inserting the free energy into the differential equation (36) we find the equation

$$\begin{aligned} 0 \;=\; \sum_i \Bigg[& x_i^2\left\{\left(\frac{\partial G_\nu}{\partial x_i}\right)^2 + \frac{1}{N_f}\frac{\partial^2}{\partial x_i^2} G_\nu\right\} \\ & + x_i\left\{\frac{2}{\beta} + \frac{\delta_\beta + \nu}{N_f}\right\}\frac{\partial}{\partial x_i}G_\nu \\ & + \frac{2}{\beta N_f}\sum_{\substack{j\\ j\neq i}}\frac{x_i x_j}{x_i - x_j}\left(\frac{\partial G_\nu}{\partial x_i} - \frac{\partial G_\nu}{\partial x_j}\right) - \frac{1}{4}x_i\Bigg]. \end{aligned} \tag{48}$$

In refs. [10, 11, 14] the classical solution for $\beta = 2$ with $\nu = 0$ was found to be

$$\sum_i \sqrt{x_i} - \frac{1}{2N_f}\sum_{i,j}\ln(\sqrt{x_i} + \sqrt{x_j}). \tag{49}$$

Ignoring the second derivative and δ_β, both of which are suppressed by an order of N_f, we likewise find the solution to (48) to lowest order in $1/N_f$

$$\sum_i \sqrt{x_i} - \frac{1}{\beta N_f}\sum_{i,j}\ln(\sqrt{x_i} + \sqrt{x_j}). \tag{50}$$

From studies of the partition function for small N_f and with equal masses, using the analytical expressions of ref. [22] we know that (50) is too simple; we expect that δ_β has to be included in the prefactor. We take care of this by expanding the free energy as

$$G_\nu \;=\; G_\nu^{(0)} + \frac{1}{N_f}G_\nu^{(1)} + \ldots \tag{51}$$

where $G_\nu^{(0)}$ is defined by (50). Inserting this expansion in (48) we now find the following differential equation for $G_\nu^{(1)}$

$$\begin{aligned} 0 \;=\; \frac{1}{N_f}\sum_i \Bigg[& \sqrt{x_i}\frac{\delta_\beta + 2\nu}{4} - \frac{\delta_\beta}{2\beta N_f}\sum_j \frac{\sqrt{x_i x_j}}{(\sqrt{x_i} + \sqrt{x_j})^2} \\ & - \frac{\nu}{\beta N_f}\sum_j \frac{\sqrt{x_i}}{\sqrt{x_i} + \sqrt{x_j}} \\ & + x_i^{3/2}\frac{\partial}{\partial x_i}G_\nu^{(1)} + \frac{2}{\beta N_f}\sqrt{x_i}\sum_j \frac{\sqrt{x_i x_j}}{\sqrt{x_i} + \sqrt{x_j}}\frac{\partial}{\partial x_i}G_\nu^{(1)} \\ & + \frac{2}{\beta N_f}\sum_{\substack{j\\ j\neq i}}\frac{x_i x_j}{x_i - x_j}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)G_\nu^{(1)}\Bigg], \end{aligned} \tag{52}$$

where we note that $G_\nu^{(1)}$ vanishes for $\beta = 2$ with vanishing topological charge since $\delta_2 = 0$.

At this point a few comments are in order. All terms in (52) have to cancel order by order in x, $x \in x_i, i = 1, \ldots, N_f$. Thus $\partial_i G_\nu^{(1)}$ only has terms of order less than or equal to $\mathcal{O}(x^{-1})$. Otherwise cancellations would only happen if $\partial_i G_\nu^{(1)}$ had terms of all positive orders of x. From this consideration it is clear that the term of order $\sqrt{x}$ in (52) should be cancelled by the term of order $x^{3/2}\partial_i G_\nu^{(1)}$. This provides a condition for $G_\nu^{(1)}$ which should then be inserted in (52). If this expression fails to cancel the remaining terms we conclude that $\partial_i G_\nu^{(1)}$ also has an $x^{-3/2}$ part and the process starts over again. Hopefully the process terminates after a finite number of iterations.

From these considerations we conclude that

$$\begin{aligned} \frac{\partial}{\partial x_i} G_\nu^{(1)} &= -\frac{\delta_\beta + 2\nu}{4x_i} + \ldots \\ \Rightarrow \qquad G_\nu^{(1)} &= -\frac{\delta_\beta + 2\nu}{4} \sum_j \ln x_j + \ldots, \end{aligned} \tag{53}$$

where in the last step we used the fact that $G_\nu^{(1)}$ should be a symmetric function of the x_i. Unfortunately the iteration process does not stop within the first three levels of iteration. Indeed, the results become increasingly complicated - thus we choose to stop the iteration at this point, using (53) as our only correction to (50). It turns out that this is enough to calculate the partition function if one requires the fermions to have equal masses. However, since all remaining terms are proportional to δ_β, (53) is precisely sufficient to calculate Virasoro constraints in the $\beta = 2$ case with non-vanishing topological charge. Furthermore, the remaining terms are independent of ν. Thus, in all classes of symmetry breaking, to this order the ν dependence of the free energy is simply given by eq. (53).

First we consider $\beta = 2$ with non-vanishing topological charge. We write the partition function as

$$\begin{aligned} (\prod_i x_i^{-\nu/2}) \mathcal{Z}_\nu^{\beta=2} &= e^{N_f(G_\nu^{(0)} + \frac{1}{N_f} G_\nu^{(1)})} Y(\{t_k\}) \\ &= e^{\sum_i \sqrt{x_i}} \prod_{i,j} \frac{1}{\sqrt{\sqrt{x_i} + \sqrt{x_j}}} \\ &\quad \times \prod_l x_l^{-\nu/2} Y(\{t_k\}). \end{aligned} \tag{54}$$

In the second line we have removed the explicit N_f-dependence of the action, and returned to our original normalization. We define the set of variables relevant to a large mass expansion differently from the small-mass expansion

$$\begin{aligned} t_k &= -\frac{2^{2k+1}}{2k+1} \mathrm{Tr}\Big((\mathcal{M}\mathcal{M}^\dagger)^{-k-1/2} \Big) \\ &= -\frac{2^{2k+1}}{2k+1} \sum_i x_i^{-k-1/2}, \end{aligned} \tag{55}$$

with the normalization chosen for convenience. Inserting (54) in the governing differential equation, eq. (36) and reading off the coefficients of powers of x_i as in the small mass expansion we find the Virasoro constraints ($\nu \to \bar{\nu}$ for $\beta = 4$)

$$L_{n,\nu}^{\beta=2} Y(\{t_k\}) \;=\; 0, \qquad n > 0, \tag{56}$$

with the Virasoro operators

$$L_{0,\nu}^{\beta=2} \;=\; \sum_{k=0}^{\infty} (k+1/2) t_k \frac{\partial}{\partial t_k} + \frac{1 - 4\nu^2}{16} + \frac{\partial}{\partial t_0} \tag{57}$$

$$\begin{aligned} L_{n,\nu}^{\beta=2} &= \sum_{k=0}^{\infty} (k+1/2) t_k \frac{\partial}{\partial t_{k+n}} + \frac{1}{4} \sum_{k=1}^{n} \frac{\partial^2}{\partial t_{k-1} \partial t_{n-k}} \\ &\quad + \frac{\partial}{\partial t_n}, \qquad n \geq 1. \end{aligned} \tag{58}$$

Again these operators satisfy the Virasoro algebra without central charge. Uniqueness of the solution is ensured by the boundary condition $Y(\{t_k\}) = 1$ for all expansion coefficients vanishing (the limit of infinite fermion masses). Comparing these Virasoro operators with the Virasoro operators obtained in ref. [10] we see that the inclusion of a non-vanishing topological charge only changes the Virasoro operator $L_0^{\beta=2}$. However, this still has profound consequences for the power expansion of the partition function, as can be readily checked.

To determine the first corrections due to including a non-vanishing topological charge, we proceed by expanding the function Y as

$$Y(\{t_k\}) = 1 + \sum_{k=0}^{\infty} c_k t_k + \sum_{0 \leq k_1 \leq k_2} c_{k_1,k_2} t_{k_1} t_{k_2} + \ldots . \tag{59}$$

The coefficients are easily obtained from the Virasoro constraints and we find to fourth order in the inverse masses

$$\begin{aligned} \mathcal{Z}_\nu^{\beta=2} &= e^{\sum_i \sqrt{x_i}} \prod_{i,j} \frac{1}{\sqrt{\sqrt{x_i} + \sqrt{x_j}}} \\ &\times \Bigg[1 + \frac{1 - 4\nu^2}{8} \mathrm{Tr}(\mathcal{M}\mathcal{M}^\dagger)^{-1/2} \\ &+ \frac{(1 - 4\nu^2)(9 - 4\nu^2)}{128} (\mathrm{Tr}(\mathcal{M}\mathcal{M}^\dagger)^{-1/2})^2 \\ &+ \frac{(1 - 4\nu^2)(9 - 4\nu^2)(17 - 4\nu^2)}{3 \cdot 1024} (\mathrm{Tr}(\mathcal{M}\mathcal{M}^\dagger)^{-1/2})^3 \\ &+ \frac{(1 - 4\nu^2)(9 - 4\nu^2)}{3 \cdot 128} \mathrm{Tr}(\mathcal{M}\mathcal{M}^\dagger)^{-3/2} \\ &+ \frac{(1 - 4\nu^2)(9 - 4\nu^2)(17 - 4\nu^2)(25 - 4\nu^2)}{3 \cdot 32768} \\ &\quad \times (\mathrm{Tr}(\mathcal{M}\mathcal{M}^\dagger)^{-1/2})^4 \\ &+ \frac{(1 - 4\nu^2)(9 - 4\nu^2)(25 - 4\nu^2)}{3 \cdot 1024} \\ &\quad \times \mathrm{Tr}(\mathcal{M}\mathcal{M}^\dagger)^{-1/2} \mathrm{Tr}(\mathcal{M}\mathcal{M}^\dagger)^{-3/2} + \ldots \Bigg] . \end{aligned} \tag{60}$$

The partition function for $\beta = 2$ is known in closed form to be [23]

$$\mathcal{Z}_\nu^{\beta=2} \;=\; \frac{\det A(\{\sqrt{x}\})}{\Delta(\{x\})}. \tag{61}$$

A is an $N_f \times N_f$ matrix defined by

$$A(\{\sqrt{x}\})_{ij} \;=\; x_i^{(j-1)/2} I_{\nu+j-1}(\sqrt{x_i}), \tag{62}$$

with I_n a modified Bessel function. Δ is the Vandermonde determinant

$$\begin{aligned}\Delta(\{x\}) \;&\equiv\; \prod_{i>j}^{N_f}(x_i - x_j)\\ &=\; \det[(x_i)^{j-1}].\end{aligned} \tag{63}$$

Expanding (61) for $N_f = 2$, which is sufficient to reveal most subtleties, we find complete agreement between this closed expression and our expansion, equation (60).

Turning now to the other classes of chiral symmetry breaking, we find the partition function for equal fermion masses. This is performed as before, by writing $\mathcal{Z}_\nu^\beta$ as the prefactor times a power expansion in the masses, that is

$$\mathcal{Z}_\nu^\beta = e^{\sum_i \sqrt{x}_i} \prod_{i,j} \frac{1}{(\sqrt{x}_i + \sqrt{x}_j)^{1/\beta}} \prod_k x_k^{-\delta_\beta/4} Y(\{t_k\}). \tag{64}$$

Inserting this expansion in (36) and setting all fermion masses equal, after some algebra, we obtain

$$-2c_0 \;=\; \frac{1 + (N_f - 1)2\delta_\beta/\beta - 4\nu^2}{8}, \tag{65}$$

where, as expected, we find that the partition functions are equal for $N_f = 1$, a feature which should be present not only to this order in the mass expansion but to all orders. We have here returned to our original normalization, removing the explicit N_f dependence from the partition function. Seeing how the ν dependence of the free energy was found to be very simple to the order examined, it should not come as a complete surprise that the ν-dependence is independent of δ_β, apart from the difference in prefactors. Explicitly writing the partition function for equal masses to lowest order (defining $x \equiv x_1 = \cdots = x_{N_f}$)

$$\begin{aligned}\mathcal{Z}_\nu^\beta \;=\;& e^{N_f\sqrt{x}} x^{-N_f^2/2\beta} x^{-N_f\delta_\beta/4}\\ &\times\left(1 + \frac{1 + (N_f - 1)2\delta_\beta/\beta - 4\nu^2}{8}\frac{N_f}{\sqrt{x}} + \ldots\right),\end{aligned} \tag{66}$$

where we as usual let $\nu \to \bar{\nu}$ for $\beta = 4$.

This result is just what we need to calculate the finite volume quenched chiral condensate in the large mass limit. To this end we replace N_f with N_v or, formally, extend the partition function with one replica set of N_v fermions and then let $N_f \to 0$. Changing variables to $\mu_v \equiv \sqrt{x}$ (with the equal fermion masses given by $m_v = \mu_v/\Sigma_0 V$) eq. (7) can also be written

$$\frac{\Sigma_\nu^\beta(\mu_v)}{\Sigma_0} \;=\; \lim_{N_v\to 0} \frac{1}{N_v}\frac{\partial}{\partial\mu_v} \ln \mathcal{Z}_\nu^\beta. \tag{67}$$

We thus calculate the quenched chiral condensate in sectors of topological charge ν ($\nu \to \bar{\nu}$ for $\beta = 4$) and in all classes of chiral symmetry breaking

$$\left.\frac{\Sigma_\nu^\beta(\mu_v)}{\Sigma_0}\right|_{\text{quenched}} = 1 - \frac{\delta_\beta}{2\mu_v} - \frac{1 - 2\delta_\beta/\beta - 4\nu^2}{8\mu_v^2} + \ldots \tag{68}$$

While the quenched chiral condensate has been previously calculated in the case of $\beta = 2$ [17], interestingly for $\beta = 1, 4$ our results show that we have to include a term of order $\mathcal{O}(\mu_v^{-1})$ in these classes. This feature turns out to be very important in the matching of the chiral condensate in the two finite volume perturbation schemes considered in the next section.

IV. QUENCHED FINITE VOLUME CHIRAL CONDENSATES

So far everything we have calculated has been in the large, albeit finite, volume, in which the usual momentum expansion is applicable. Now we turn to the other finite volume regime, in which the correlation lengths of the Goldstone modes are much larger than the volume of the box. In this regime the usual p-expansion of chiral perturbation theory breaks down due to the propagation of zero-momentum Goldstone bosons. Thus another expansion scheme, known as the ϵ-expansion [15, 24], is required. In ref. [17] it was shown, in the case of $\beta = 2$, that a region exists in which the two expansion schemes overlap. The main result of this section is that matching is not fortuitous, the result carries over to the $\beta = 1, 4$ cases. We demonstrate this by calculating the quenched chiral condensate in the ϵ-expansion and comparing with the corresponding result (28) of the p-expansion. Thus we conclude that matching of the two expansion schemes is possible in all classes of chiral symmetry breaking.

Again we use the $\mathcal{O}(p^2)$ Lagrangian which we parameterize as in (2), i.e.

$$\begin{aligned}\mathcal{L}_{\text{eff}} \;=\;& \frac{F_\beta^2}{4}\text{Tr}(\partial_\mu U^\dagger \partial_\mu U) - \frac{\Sigma_0}{2}\text{Tr}(\mathcal{M}^{(\beta)}(U + U^\dagger))\\ &+ \frac{m_0^2}{2N_c}\Phi_0^2 + \frac{\alpha}{2N_c}\partial_\mu\Phi_0\partial_\mu\Phi_0.\end{aligned} \tag{69}$$

The virtue of the ϵ-expansion is that it counts zero-momentum modes as order $\mathcal{O}(1)$ while non-zero modes count as higher orders in ϵ. Letting the four-volume be $V \equiv L^3/T$ and treating the fermion masses as small in comparison with T and $1/L$ one fixes the counting of orders of ϵ by setting $1/L = \mathcal{O}(\epsilon)$. The fermion masses m_i are then counted as $\mathcal{O}(\epsilon^4)$ which implies $M_{ij} \sim \mathcal{O}(\epsilon^2)$. Thus all graphs which exclusively involve zero momentum propagators are counted as being of order $\mathcal{O}(\epsilon^0)$ [15], see for instance eq. (10). In the unquenched theory, adding a mass term for the non-zero modes provides a Gaussian damping factor for these modes which ensures that the non-zero modes count as $\mathcal{O}(\epsilon^1)$ [15]. In the quenched theory the counting is not as simple since the mass term m_0 is a free parameter. But following [17, 25] we perform a simultaneous expansion in ϵ and $m_0^2/(N_cF^2)$, as a way of avoiding having to evaluate all orders of m_0 for each order of ϵ.

The idea, adopted from ref. [15], is to collect the zero-momentum modes in the constant matrix $U_0 \in G/H$ and write

$$U(x) \;=\; U_0 e^{i\sqrt{2}\xi(x)/F_\beta}, \tag{70}$$

where the field $\xi(x) \in G/H$ contains the non-zero momentum modes. For the non-zero modes we choose the same parameterization as in (8), for $\beta = 1$, and (17), for $\beta = 4$. That is ($\beta = 1$)

$$\xi = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi & \psi \\ \psi^\dagger & \phi^T \end{pmatrix}, \tag{71}$$

with ϕ Hermitian and ψ anti-symmetric complex fields, and ($\beta = 4$)

$$\xi = \begin{pmatrix} A_{11} & \frac{1}{\sqrt{2}}A_{12} & \cdots & \frac{1}{\sqrt{2}}A_{1,N_f+N_v} \\ \frac{1}{\sqrt{2}}A_{12} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{1}{\sqrt{2}}A_{1,N_f+N_v} & \cdots & \cdots & A_{N_f+N_v,N_f+N_v} \end{pmatrix}, \tag{72}$$

with ξ being a real, symmetric matrix.

We now wish to perform this change of variables and integrate over the zero modes and the non-zero momentum modes separately. Thus we have to perform the change of variables (70) in the partition function. The Jacobian of this change of variables is, to lowest order, just a constant, corresponding to a shift of the vacuum energy. Noting that integrals of odd powers of ξ vanish, with this change of variables the lowest order action becomes

$$\begin{aligned} S \;=\; & \int d^4x \mathrm{Tr}\left[\frac{1}{2}\partial_\mu\xi(x)\partial_\mu\xi(x) \right.\\ & \left. +\frac{m_0^2}{2N_c}\left(\Xi_0 + \Xi(x)\right)^2 + \frac{\alpha}{2N_c}\partial_\mu\Xi(x)\partial_\mu\Xi(x)\right] \\ & -\frac{\Sigma_0}{2}\mathrm{Tr}\left[\mathcal{M}(U_0 + U_0^\dagger)\left(V - \frac{1}{F_\beta^2}\int dx\xi^2\right)\right], \end{aligned} \tag{73}$$

where $U_0 \equiv e^{i\sqrt{2}\varphi_0/F_\beta}$, and $\Xi_0 \equiv \mathrm{Tr}\varphi_0$. The mass matrices $\mathcal{M}$ are defined as in eqs. (9) and (18).

Since we are once again interested in calculating the one loop correction to the chiral condensate we integrate out the non-zero momentum modes, $\xi(x)$. To this end we define the non-zero momentum modes

$$\bar{\Delta}_{ij} \;\equiv\; \frac{1}{V}\sum_{p\neq 0}\frac{1}{p^2 + M_{ij}^2} \tag{74}$$

$$\bar{G}_{ij} \;\equiv\; \frac{1}{V}\sum_{p\neq 0} G(M_{ij}^2). \tag{75}$$

The two-point functions for the off-diagonal mesons are related to the full propagators by

$$\bar{\Delta}_{ij} \;=\; \Delta_{ij} - \frac{1}{VM_{ij}^2}. \tag{76}$$

The propagator of the diagonal mesons, $G(M_{ij}^2)$, is defined as in eq. (25).

Thus integrating out the fluctuating fields we find the fully quenched one loop correction to the effective Lagrangian for the zero momentum modes by putting $N_f \to 0$,

$$\begin{aligned} & \lim_{N_v\to 0}\frac{1}{N_v}\left\langle \frac{\Sigma_0}{2F_\beta^2}\mathrm{Tr}\left[\mathcal{M}(U_0 + U_0^\dagger)\left(\int dx\xi(x)^2\right)\right]\right\rangle \\ = \; & \lim_{N_v\to 0}\frac{1}{N_v}\frac{V\Sigma_0}{2F^2}\mathrm{Tr}(\mathcal{M}(U_0+U_0^\dagger)) \\ & \times\left(\frac{2}{\beta}(N_v - 1)\bar{\Delta}_{vv} + \bar{G}_{vv}\right) \\ = \; & \lim_{N_v\to 0}\frac{1}{N_v}\frac{V\Sigma_0}{2F^2}\mathrm{Tr}(\mathcal{M}(U_0+U_0^\dagger)) \\ & \times\left(\delta_\beta\bar{\Delta}_{vv} - (1+\delta_{\beta,1})\frac{\alpha}{N_c}\bar{\Delta}_{vv} + \right. \\ & \left. \frac{1}{N_c}(1+\delta_{\beta,1})(m_0^2 - \alpha M_{vv}^2)\partial_{M_{vv}^2}\bar{\Delta}_{vv}\right). \end{aligned} \tag{77}$$

In the last equality we have made use of the fact that the first part of the diagonal propagator is the same as for the off-diagonal propagator. That is

$$\bar{G}_{ij} = \bar{\Delta}_{ij} - \sum_{p\neq 0}\frac{(\delta_{\beta,1}+1)(m_0^2 + \alpha p^2)/N_c}{(p^2 + M_{ii}^2)(p^2+M_{jj}^2)\mathcal{F}^\beta(p^2)}. \tag{78}$$

Equation (77) is very useful, since from the correction to the zero'th order approximation we see that the one-loop improved effective partition function can be written in the same way as the zero'th order approximation

$$\mathcal{Z}^\beta \;=\; \int_{G/H} dU_0 e^{\frac{1}{2}V\Sigma_0\mathrm{Tr}\mathcal{M}(U_0+U_0^\dagger) - \frac{Vm_0^2}{2N_c}\Xi_0^2}, \tag{79}$$

by simply changing the mass eigenvalues into

$$\begin{aligned} \mu_v' \;=\; \mu_v & \left[1 - \frac{1}{F^2N_c}\left\{N_c\delta_\beta\bar{\Delta}_{vv} - (1+\delta_{\beta,1})\alpha\bar{\Delta}_{vv}\right.\right. \\ & \left.\left. +(1+\delta_{\beta,1})(m_0^2 - \alpha M_{vv}^2)\partial_{M_{vv}^2}\bar{\Delta}_{vv}\right\}\right]. \end{aligned} \tag{80}$$

Notice that we again define $\mu_v \equiv m_vV\Sigma_0$.

Thus we can exploit the usual formula, (7), to find the one loop improved quenched chiral condensate

$$\frac{\Sigma(\mu_v)}{\Sigma_0} \;=\; \lim_{N_v\to 0}\frac{1}{N_v}\frac{\partial\mu_v'}{\partial\mu_v}\frac{\partial}{\partial\mu_v'}\ln\mathcal{Z}^\beta(\mu_v'). \tag{81}$$

Now we make use of our calculation of the chiral condensate in sectors of fixed topological charge. When calculating the chiral condensate, we have to sum over topology [26]. To perform this summation we follow the idea of ref. [17] and include an arbitrary vacuum angle in the partition function by absorbing it into the field Ξ_0. Next we perform a Fourier transform projecting the vacuum angle onto sectors of topological charge.

Identification of the topological susceptibility [20] is performed in Appendix B. The result is

$$\left\langle \nu^2 \right\rangle \;=\; (1+\delta_{\beta,1})\frac{m_0^2 V F^2}{2N_c}. \tag{82}$$

The summation over topologies is

$$\Sigma^{\beta}(\mu_v) = \left\langle \Sigma_\nu^{\beta}(\mu_v) \right\rangle = \sum_{\nu=-\infty}^{\infty} \frac{\mathcal{Z}_\nu^{\beta}(\mu_v)\Sigma_\nu^{\beta}(\mu_v)}{\mathcal{Z}^{\beta}(\mu_v)}. \tag{83}$$

Inserting the quenched chiral condensate in sectors of topological charge calculated in eq. (68), to the order considered we find

$$\begin{aligned}\frac{\Sigma^{\beta}(\mu_v)}{\Sigma_0} \;&=\; \left\langle \frac{\Sigma_\nu^{\beta}(\mu_v)}{\Sigma_0} \right\rangle \\ &=\; 1 - \frac{\delta_\beta}{2\mu_v} - \frac{1 - 2\delta_\beta/\beta - 4\left\langle \nu^2 \right\rangle}{8\mu_v^2}. \end{aligned} \tag{84}$$

Now we look for a size of the volume V in which the two finite volume regimes match [17]. Obviously (84) cannot be a simple perturbation of the infinite volume theory, since $\mu_v \equiv m_v V \Sigma_0$ in this regime is finite while $\left\langle \nu^2 \right\rangle \sim V$. Thus the requirement that the last term in (84) is a small perturbation is

$$V \;\gg\; (1+\delta_{\beta,1})\frac{F^2 m_0^2}{4N_c m_v^2 \Sigma_0^2}. \tag{85}$$

The requirement of this extreme finite volume regime is that $V \ll M_{vv}^{-4} = F^4/4m_v^2\Sigma_0^2$. Both limits can be met by letting $N_c \to \infty$ while keeping all other quantities fixed, since then $F^2 \gg (1+\delta_{\beta,1})m_0^2/N_c$.

Having established a region in which a match is possible, let us first review the result (28) for the chiral condensate in the large, but finite, volume. Rewriting this for the quenched limit we obtain

$$\begin{aligned}\frac{\Sigma^{\beta}(\mu_v)}{\Sigma_0} \;&=\; 1 - \frac{1}{F^2 N_c}\left\{ N_c \delta_\beta \Delta_{vv} - (1+\delta_{\beta,1})\alpha\Delta_{vv} \right. \\ &\qquad \left. +(1+\delta_{\beta,1})(m_0^2 - \alpha M_{vv}^2)\partial_{M_{vv}^2}\Delta_{vv} \right\}. \end{aligned} \tag{86}$$

In the extreme finite volume we also have all the ingredients necessary for calculating the quenched chiral condensate. Inserting eqs. (80) and (84) in eq. (81) we find to leading order

$$\begin{aligned}\frac{\Sigma^{\beta}(\mu_v)}{\Sigma_0} \;&=\; \left(1 - \frac{\delta_\beta}{VF^2M_{vv}^2} + (1+\delta_{\beta,1})\frac{m_0^2}{N_c F^2 M_{vv}^4 V}\right) \\ &\quad \times \left[1 - \frac{1}{F^2N_c}\left\{ N_c\delta_\beta \bar{\Delta}_{vv} - (1+\delta_{\beta,1})\alpha\bar{\Delta}_{vv} \right.\right. \\ &\qquad \left.\left. +(1+\delta_{\beta,1})(m_0^2 - \alpha M_{vv}^2)\partial_{M_{vv}^2}\bar{\Delta}_{vv} \right\} \right] \\ &=\; 1 - \frac{1}{F^2N_c}\left\{ N_c\delta_\beta\Delta_{vv} - (1+\delta_{\beta,1})\alpha\Delta_{vv} \right. \\ &\qquad \left. +(1+\delta_{\beta,1})(m_0^2 - \alpha M_{vv}^2)\partial_{M_{vv}^2}\Delta_{vv} \right\}. \end{aligned} \tag{87}$$

In this calculation, we have made use of the fact that $\left\langle \nu^2 \right\rangle \gg 1$ while $\mu_v \sim \mathcal{O}(1)$.

Comparing the results (87) for the extreme finite volume and (86) for the ordinary finite volume, we find the promised overlap of the two perturbative schemes. This is an important consistency check on our calculations and it is a very attractive feature of this calculation that it is precisely the term proportional to $1/\mu_v$ in the quenched chiral condensate, absent in the $\beta = 2$ case, which makes this matching possible.

V. CONCLUSIONS

We have analyzed the three patterns of chiral symmetry breaking relevant to chiral perturbation theory, in particular $\beta = 1$ and $\beta = 4$ corresponding to fermions in a pseudo-real and real representation, respectively.

In the usual p-expansion of chiral perturbation theory, we have demonstrated how the replica method is equivalent to the supersymmetric method by comparing the Feynman rules to one loop order for the $\beta = 1, 4$ classes of chiral symmetry breaking, and seeing how the calculations proceed it is fairly obvious how this generalizes to higher order loop calculations.

Next, using Schwinger-Dyson equations, we have obtained a governing Schwinger-Dyson equation for the partition function in all classes of chiral symmetry breaking and in sectors of fixed topological charge. We have demonstrated how it is possible to write this differential equation in terms of the mass eigenvalues only, with all dependence upon the class of chiral symmetry breaking being through the Dyson index, β.

We have utilized the governing differential equation to determine Virasoro constraints in the small mass expansion, which exactly matches those derived in ref. [7]. We have also calculated Virasoro constraints in the large mass expansion of the $\beta = 2$ partition function in sectors of fixed topological charge. We note how the inclusion of the topological charge in the Virasoro constraints becomes possible by simply expanding the saddle-point contribution of the free energy to next to leading order in $1/N_f$ instead of only keeping leading order terms.

We find indications of how the next to leading order of the saddle-point approximation for $\beta = 1, 4$ seems to consist of an infinite series in the fermion masses. However, we note that it is still possible to find the first terms in a power expansion of the partition function in these classes of chiral symmetry breaking if one puts the masses equal. Having thus obtained the equal-mass partition function in sectors of fixed topological charge to the order needed, we calculate the quenched chiral condensate. We find that the mass-dependence of the quenched chiral condensate is different for real and pseudo-real representations of the fermions compared with the formerly known result for fermions in complex representations.

In the other relevant finite volume, the volume smaller than the correlation lengths of the Goldstone bosons, we utilize the ϵ-expansion to calculate the first correction to the quenched chiral condensate. It turns out that it is exactly the above mentioned difference in mass-dependence of $\beta = 1, 4$ which makes the quenched chiral condensate of this finite volume match the corresponding quenched chiral condensate of the ordinary p-expansion. Thus we find a region in which the two perturbative expansions of chiral perturbation theory, the ϵ-expansion and the p-expansion, precisely match.

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APPENDIX A: SCHWINGER-DYSON EQUATIONS

In this appendix we will find Schwinger-Dyson equations in all three classes of chiral symmetry breaking. Theories with complex fermions, such as ordinary QCD, have been considered before [14], but since the calculations here are basically a less complicated version of the calculations in the $\beta = 1, 4$ cases, we will line them up. Also, this will be done in a somewhat different way from the literature.

To avoid an unnecessarily complicated notation we consider first theories with vanishing topological charge. The topological charge is easily included at the end of the calculations.

1. $\beta = 2$

We start out with the partition function in sectors of vanishing topological charge

$$\mathcal{Z}^{\beta=2} = \int_{U\in U(N_f)} dU\, e^{\frac{1}{2}\mathrm{Tr}\left(\mathcal{M}U^\dagger + \mathcal{M}^\dagger U\right)}. \quad \text{(A1)}$$

As discussed in section III the crucial property of this partition function is that it only depends on the N_f eigenvalues x_i of $\mathcal{M}\mathcal{M}^\dagger$. To arrive at the Schwinger-Dyson equations, we utilize the fact that the integral of a total derivative vanishes. Thus

$$\begin{aligned} 0 &= \int_{U\in U(N_f)} dU \mathrm{Tr} t^a \nabla^a \left(F(U) e^{\frac{1}{2}\left(\mathcal{M}U^\dagger + \mathcal{M}^\dagger U\right)}\right) \\ &= \Big\langle \mathrm{Tr} t^a \nabla^a F(U) \\ &\quad + \frac{1}{2}\mathrm{Tr}(t^a F(U)) \nabla^a \mathrm{Tr}\left(\mathcal{M}U^\dagger + \mathcal{M}^\dagger U\right)\Big\rangle. \quad \text{(A2)} \end{aligned}$$

The left derivative ∇^a on $U(N_f)$ is defined through the relations $\nabla^a U = it^a U$ and $\nabla^a U^\dagger = -iU^\dagger t^a$ where t^a are the generators of $U(N_f)$. A simple consistency check on these is that $\nabla^a U U^\dagger = 0$. An explicit representation of ∇^a is

$$\nabla^a = i(t^a U)_{ij} \frac{\partial}{\partial U_{ij}}. \quad \text{(A3)}$$

We normalize the generators of the unitary group according to

$$t^a_{ij} t^a_{kl} = \frac{1}{2}\delta_{il}\delta_{jk}. \quad \text{(A4)}$$

Getting a useful Schwinger-Dyson equation is now a matter of choosing the function $F(U)$ in a clever way. It turns out that with the choice of $F(U) = U\mathcal{M}^\dagger$ we arrive at the equation considered in [14]

$$N_f \left\langle \mathrm{Tr}\mathcal{M}^\dagger U\right\rangle = \frac{1}{2}\left\langle \mathrm{Tr}\mathcal{M}\mathcal{M}^\dagger - \mathrm{Tr}U\mathcal{M}^\dagger U\mathcal{M}^\dagger\right\rangle, \quad \text{(A5)}$$

where we have used (A4).

Now we follow the idea of ref. [14]. We know that

$$\left\langle \mathrm{Tr}\mathcal{M}^\dagger U\right\rangle = 2\left\langle \mathcal{M}^\dagger_{ij}\frac{\partial}{\partial \mathcal{M}^\dagger_{ij}}\right\rangle \equiv 2\left\langle D_1\right\rangle, \quad \text{(A6)}$$

$$\begin{aligned} \left\langle \mathrm{Tr}U\mathcal{M}^\dagger U\mathcal{M}^\dagger\right\rangle &= 4\left\langle \mathcal{M}^\dagger_{ij}\mathcal{M}^\dagger_{kl}\frac{\partial}{\partial \mathcal{M}^\dagger_{kj}}\frac{\partial}{\partial \mathcal{M}^\dagger_{il}}\right\rangle \\ &\equiv 4\left\langle D_2\right\rangle. \quad \text{(A7)} \end{aligned}$$

To derive how the differential operators D_1 and D_2 act on the partition function in terms of the eigenvalues we use the fact that $\mathcal{Z}^{\beta=2}$ only depends on the N_f functions

$$\begin{aligned} \phi_p &\equiv \mathrm{Tr}((\mathcal{M}\mathcal{M}^\dagger)^p) \\ &= \sum_i x_i^p, \quad \text{(A8)} \end{aligned}$$

$p = 1, \ldots, N_f$. This follows from the fact that $\mathcal{Z}^{\beta=2}$ depends only on the eigenvalues as well as on permutation symmetry among these. Using the chain rule we find

$$D_1 \mathcal{Z}^{\beta=2} = \sum_p \frac{\partial \mathcal{Z}^{\beta=2}}{\partial \phi_p} D_1 \phi_p, \quad \text{(A9)}$$

so we only need to know how D_1 operates on the basis functions, i.e.

$$\begin{aligned} D_1\phi_p &= p\mathcal{M}^\dagger_{ij}((\mathcal{M}\mathcal{M}^\dagger)^{p-1}\mathcal{M})_{ji} \quad \text{(A10)} \\ &= p\phi_p. \quad \text{(A11)} \end{aligned}$$

Thus we see that

$$D_1 = x_i \frac{\partial}{\partial x_i}. \quad \text{(A12)}$$

In the same way

$$\begin{aligned} D_2\phi_p &= \sum_{p'=1}^{p} p\phi_{p'}\phi_{p-p'} \\ &= p(p-1)\sum_i x_i^p \\ &\quad + \sum_{\substack{i,j \\ i\neq j}} \frac{x_i x_j}{x_i - x_j}(p x_i^{p-1} + p x_j^{p-1}), \quad \text{(A13)} \end{aligned}$$

and following [14] we see that the second derivative is equivalent to

$$D_2 \;=\; x_i^2 \frac{\partial^2}{\partial x_i^2} + \sum_{\substack{i,j \\ i\neq j}} \frac{x_i x_j}{x_i - x_j}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right). \quad \text{(A14)}$$

As noted in [14] this procedure needs further justification. From the invariance properties of the partition function we know that $\mathcal{Z}^{\beta=2}$is a function of the basis functions only. Accordingly we should be careful when concluding which form the double differential operator takes in terms of the eigenvalues. Expanding the partition function in terms of the basis functions we see that in principle things could become complicated from the chain rule when D_2 acts on a product of the basis functions. We can check that this is not the case by letting our "guess" (A14) act on a product $\phi_p\phi_q$ and comparing this with the result obtained by letting the known correct matrix form of the derivative (A7) act on $\phi_p\phi_q$. Since we are dealing with a double derivative no new complications arise if we look at a term $\phi_p\phi_q\phi_r\ldots$ in the expansion of the partition function. This test, in fact, gives the correct result.

Using (A5) we collect the terms and find the following Schwinger-Dyson equation for QCD with fermions in a complex representation:

$$\begin{aligned}&\left[x_i^2\partial_i^2 + N_f x_i\partial_i + \sum_{\substack{i,j \\ i\neq j}} \frac{x_i x_j}{x_i - x_j}\left(\partial_i - \partial_j\right)\right]\mathcal{Z}^{\beta=2}\\ &= \frac{1}{4}\sum_i x_i \mathcal{Z}^{\beta=2}. \qquad \text{(A15)}\end{aligned}$$

2. $\beta = 1$

In the case of fermions in a pseudo-real representation the Goldstone bosons live in the coset $SU(2N_f)/Sp(2N_f)$. This coset can be parametrized by UIU^t, where I is the antisymmetric $2N_f \times 2N_f$ unit matrix

$$I \;=\; \begin{bmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{bmatrix},$$

and $U \in U(2N_f)$. The partition function with vanishing topological charge is [7]

$$\mathcal{Z}^{\beta=1} \;=\; \int_{U\in U(2N_f)} dU e^{\frac{1}{4}\mathrm{Tr}\left(\tilde{\mathcal{M}}^\dagger UIU^t + \tilde{\mathcal{M}}(UIU^t)^\dagger\right)}, \quad \text{(A16)}$$

where the mass matrix $\tilde{\mathcal{M}}$ is an arbitrary complex antisymmetric matrix. Note also that the combination $UIU^t \equiv V$ is antisymmetric, which will turn out to be very useful in the following. Again, the partition function is a function of the eigenvalues of $\tilde{\mathcal{M}}\tilde{\mathcal{M}}^\dagger$ and since the Goldstone modes are parameterized by the unitary group we can follow the same approach as in the case of $\beta = 2$. We know that

$$0 = \int_{U\in U(2N_f)} dU \mathrm{Tr} t^a \nabla^a\Big(F(U) e^{\frac{1}{4}\mathrm{Tr}\left(\tilde{\mathcal{M}}^\dagger V + \tilde{\mathcal{M}} V^\dagger\right)}\Big), \quad \text{(A17)}$$

and if we, inspired by the $\beta = 2$ calculation, choose $F(U) = V\tilde{\mathcal{M}}^\dagger$ this results in

$$\begin{aligned}\Big\langle (2N_f - 1)\mathrm{Tr}\tilde{\mathcal{M}}^\dagger V\Big\rangle \;&=\; \frac{1}{2}\Big\langle \mathrm{Tr}\tilde{\mathcal{M}}\tilde{\mathcal{M}}^\dagger \\ &\qquad -\mathrm{Tr}V\tilde{\mathcal{M}}^\dagger V\tilde{\mathcal{M}}^\dagger\Big\rangle. \quad \text{(A18)}\end{aligned}$$

It is important to realize that since both V and $\mathcal{M}^\dagger$ are antisymmetric, this equation should be treated a bit differently from the $\beta = 2$ case. Being careful, we define the differential operators

$$\begin{aligned}\Big\langle \mathrm{Tr}\tilde{\mathcal{M}}^\dagger V\Big\rangle \;&=\; 2\left\langle \sum_{\substack{i,j \\ i\neq j}} \tilde{\mathcal{M}}^\dagger_{ij}\frac{\partial}{\partial\tilde{\mathcal{M}}^\dagger_{ij}}\right\rangle \\ &\equiv\; 2\left\langle D_1\right\rangle, \qquad \text{(A19)}\\ \Big\langle \mathrm{Tr}V\tilde{\mathcal{M}}^\dagger V\tilde{\mathcal{M}}^\dagger\Big\rangle \;&=\; 4\left\langle \sum_{\substack{i,j,k,l \\ i\neq l, j\neq k}} \tilde{\mathcal{M}}^\dagger_{ij}\tilde{\mathcal{M}}^\dagger_{kl}\frac{\partial}{\partial\tilde{\mathcal{M}}^\dagger_{kj}}\frac{\partial}{\partial\tilde{\mathcal{M}}^\dagger_{il}}\right\rangle \\ &\equiv\; 4\left\langle D_2\right\rangle. \qquad \text{(A20)}\end{aligned}$$

Again we let these act on the basis functions $\phi_p, p = 1,\ldots,N_f$. The basis functions are defined in the same way as in the $\beta = 2$ case but differ, since the matrices in this case are $2N_f \times 2N_f$. Asuming that the theory has no di-quark terms, the mass matrix has the form

$$\tilde{\mathcal{M}} = \begin{bmatrix} 0 & \mathcal{M} \\ -\mathcal{M}^t & 0 \end{bmatrix}, \quad \text{(A21)}$$

which results in the basis functions (again letting x_i, $i = 1,\ldots,N_f$ be the eigenvalues of the matrix $\mathcal{M}\mathcal{M}^\dagger$)

$$\begin{aligned}\phi_p \;&=\; \mathrm{Tr}((\tilde{\mathcal{M}}\tilde{\mathcal{M}}^\dagger)^p) \\ &=\; 2\sum_i x_i^p. \qquad \text{(A22)}\end{aligned}$$

Though the sums in eqs. (A19) and (A20) do not run over all indices, when doing the actual derivatives the results turn out rather nicely. Indeed for the first derivative we find

$$\begin{aligned}D_1\phi_p \;&=\; \sum_{\substack{i,j \\ i\neq j}} \tilde{\mathcal{M}}^\dagger_{ij}\frac{\partial}{\partial\tilde{\mathcal{M}}^\dagger_{ij}}\mathrm{Tr}((\tilde{\mathcal{M}}\tilde{\mathcal{M}}^\dagger)^p) \\ &=\; p\sum_{\substack{i,j \\ i\neq j}} \tilde{\mathcal{M}}^\dagger_{ij}\Big[((\tilde{\mathcal{M}}\tilde{\mathcal{M}}^\dagger)^{p-1}\tilde{\mathcal{M}})_{ji} \\ &\qquad\qquad -((\tilde{\mathcal{M}}\tilde{\mathcal{M}}^\dagger)^{p-1}\tilde{\mathcal{M}})_{ij}\Big] \\ &=\; 2p\phi_p, \qquad \text{(A23)}\end{aligned}$$

from which we recognize

$$D_1 \;=\; 2x_i\frac{\partial}{\partial x_i}. \tag{A24}$$

Turning to the second order differential operator

$$\begin{aligned} D_2\phi_p \;&=\; \sum_{\substack{i,j,k,l\\ i\neq l, j\neq k}} \tilde{\mathcal{M}}^\dagger_{ij}\tilde{\mathcal{M}}^\dagger_{kl}\frac{\partial}{\partial\tilde{\mathcal{M}}^\dagger_{kj}}\frac{\partial}{\partial\tilde{\mathcal{M}}^\dagger_{il}}\mathrm{Tr}((\tilde{\mathcal{M}}\tilde{\mathcal{M}}^\dagger)^p)\\ &=\; 2p\sum_{p'=1}^{p}\phi_{p'}\phi_{p-p'} - 2p(p-1)\phi_p\\ &=\; 8p\sum_{\substack{i,j\\ i\neq j}}\frac{x_ix_j}{x_i-x_j}(x_i^{p-1}-x_j^{p-1})\\ &\quad +8p(p-1)\sum_i x_i^p - 4p(p-1)\sum_i x_i^p, \end{aligned} \tag{A25}$$

from which we conclude that

$$D_2 = 2x_i^2\frac{\partial^2}{\partial x_i^2} + 4\sum_{\substack{i,j\\ i\neq j}}\frac{x_ix_j}{x_i-x_j}\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right). \tag{A26}$$

Again we have to verify this double derivative on a product $\phi_p\phi_q$. The test produces exactly the necessary result.

Collecting the terms using eq. (A18) we find the differential equation

$$\begin{aligned}&\left[x_i^2\partial_i^2 + (2N_f-1)x_i\partial_i + 2\sum_{\substack{i,j\\ i\neq j}}\tfrac{x_ix_j}{x_i-x_j}\left(\partial_i-\partial_j\right)\right]\mathcal{Z}^{\beta=1}\\ &= \tfrac{1}{4}\sum_i x_i\mathcal{Z}^{\beta=1}.\end{aligned} \tag{A27}$$

3. $\beta = 4$

For fermions in a real representation the coset is $SU(N_f)/SO(N_f)$. Once again this can be parameterized by elements from the unitary group, in this case by UU^t. Here the mass matrix is an arbitrary symmetric complex matrix and the partition function with vanishing topological charge is [21, 22]

$$\mathcal{Z}^{\beta=4} \;=\; \int_{U\in U(N_f)} dU e^{\frac{1}{2}\mathrm{Tr}\left(\mathcal{M}^\dagger UU^t + \mathcal{M}(UU^t)^\dagger\right)}. \tag{A28}$$

Proceeding in the familiar manner, with W being the symmetric matrix UU^t, we find

$$0 = \int_{U\in U(N_f)} dU\mathrm{Tr}t^a\nabla^a\left(F(U)e^{\frac{1}{2}\mathrm{Tr}\left(\mathcal{M}^\dagger W + \mathcal{M}W^\dagger\right)}\right). \tag{A29}$$

Defining $F(U) = W\mathcal{M}^\dagger$ we find the Schwinger-Dyson equation

$$\left\langle (N_f+1)\mathrm{Tr}\mathcal{M}^\dagger W\right\rangle = \left\langle \mathrm{Tr}\mathcal{M}\mathcal{M}^\dagger - \mathrm{Tr}W\mathcal{M}^\dagger W\mathcal{M}^\dagger\right\rangle. \tag{A30}$$

Having seen the techniques in the anti-symmetric case of $\beta = 1$ this is much the same. The only complication is that in this case the diagonal elements of the mass matrix are not equal to zero. Again we carefully define the differential operators

$$\begin{aligned}\left\langle \mathrm{Tr}\mathcal{M}^\dagger W\right\rangle \;&=\; \left\langle 2\sum_i \mathcal{M}^\dagger_{ii}\frac{\partial}{\partial\mathcal{M}^\dagger_{ii}} + \sum_{\substack{i,j\\ i\neq j}}\mathcal{M}^\dagger_{ij}\frac{\partial}{\partial\mathcal{M}^\dagger_{ij}}\right\rangle\\ &\equiv\; \left\langle D_1\right\rangle,\end{aligned} \tag{A31}$$

$$\begin{aligned}\left\langle \mathrm{Tr}W\mathcal{M}^\dagger W\mathcal{M}^\dagger\right\rangle \;&=\; \Bigg\langle \sum_{\substack{i,j,k,l\\ i\neq l, j\neq k}}\mathcal{M}^\dagger_{ij}\mathcal{M}^\dagger_{kl}\frac{\partial}{\partial\mathcal{M}^\dagger_{kj}}\frac{\partial}{\partial\mathcal{M}^\dagger_{il}}\\ &\quad +2\sum_{\substack{i,j,k\\ j\neq k}}\mathcal{M}^\dagger_{ij}\mathcal{M}^\dagger_{ki}\frac{\partial}{\partial\mathcal{M}^\dagger_{kj}}\frac{\partial}{\partial\mathcal{M}^\dagger_{ii}}\\ &\quad +2\sum_{\substack{i,j,l\\ i\neq l}}\mathcal{M}^\dagger_{ij}\mathcal{M}^\dagger_{jl}\frac{\partial}{\partial\mathcal{M}^\dagger_{jj}}\frac{\partial}{\partial\mathcal{M}^\dagger_{il}}\\ &\quad + 4\sum_{i,j}\mathcal{M}^\dagger_{ij}\mathcal{M}^\dagger_{ji}\frac{\partial}{\partial\mathcal{M}^\dagger_{jj}}\frac{\partial}{\partial\mathcal{M}^\dagger_{ii}}\Bigg\rangle\\ &\equiv\; \left\langle D_2\right\rangle.\end{aligned} \tag{A32}$$

The partition function only depends on the eigenvalues of $\mathcal{M}\mathcal{M}^\dagger$ and since we have returned to working with $N_f\times N_f$ matrices the ϕ_p, $p = 1,\ldots,N_f$ are exactly the same as for $\beta = 2$. That is

$$\phi_p \;=\; \mathrm{Tr}((\mathcal{M}\mathcal{M}^\dagger)^p) = \sum_i x_i^p. \tag{A33}$$

In this case we find that

$$D_1\phi_p \;=\; 2p\phi_p, \tag{A34}$$

and thus

$$D_1 \;=\; 2x_i\frac{\partial}{\partial x_i}. \tag{A35}$$

The second derivative looks complicated but after some algebra we end up with the nice expression

$$D_2\phi_p \;=\; 2p\sum_{p'=1}^{p}\phi_{p'}\phi_{p-p'} + 2p(p-1)\phi_p, \tag{A36}$$

from which we deduce that

$$D_2 = 2\sum_{\substack{i,j\\ i\neq j}}\frac{x_ix_j}{x_i-x_j}\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right) + 4x_i^2\frac{\partial^2}{\partial x_i^2}. \tag{A37}$$

Now we are able to collect the terms and we end up with the result

$$\begin{aligned}&\left[x_i^2\partial_i^2 + \tfrac{N_f+1}{2}x_i\partial_i + \tfrac{1}{2}\sum_{\substack{i,j\\ i\neq j}}\tfrac{x_ix_j}{x_i-x_j}\left(\partial_i-\partial_j\right)\right]\mathcal{Z}^{\beta=4}\\ &= \tfrac{1}{4}\sum_i x_i\mathcal{Z}^{\beta=4}.\end{aligned} \tag{A38}$$

4. Including the topological charge

Including the topological charge in the Schwinger-Dyson equations is now quite simple. In all classes of chiral symmetry breaking the topological charge enters the partition function as a change of measure. Again we make use of the fact that the integral of a total derivative vanishes. Thus we get an extra term in the Schwinger-Dyson equations corresponding to the derivative of the change in measure.

More specifically, consider first $\beta = 2$. Allowing a non-vanishing topological charge changes the measure in (A1) by $dU \to dU(\det U)^\nu$. Thus we need to know the derivative of $\det U$

$$\begin{aligned}\nabla^a(\det U) &= i(t^a U)_{ij}\frac{\partial}{\partial U_{ij}}\det U \\ &= i(t^a U)_{ij}U^{-1}_{ji}\det U \\ &= i\mathrm{Tr}(t^a)\det U \end{aligned} \tag{A39}$$

since

$$\begin{aligned}\det(U+\delta U) &= \det(U(1+U^{-1}\delta U)) \\ &= \det U(1+\mathrm{Tr}(U^{-1}\delta U)+\ldots). \end{aligned} \tag{A40}$$

Thus we see that the expectation value equation (A5) changes into

$$(N_f+\nu)\left\langle \mathrm{Tr}\mathcal{M}^\dagger U\right\rangle = \frac{1}{2}\left\langle \mathrm{Tr}\mathcal{M}\mathcal{M}^\dagger - \mathrm{Tr}U\mathcal{M}^\dagger U\mathcal{M}^\dagger\right\rangle, \tag{A41}$$

which only alters the coefficient in front of the differential operator D_1. Modifying the partition function, of course also implies that the dependencies are changed. As discussed in section III, $(\det\mathcal{M})^{-\nu}\mathcal{Z}_\nu^{\beta=2}$ is a function of the eigenvalues of $\mathcal{M}\mathcal{M}^\dagger$ only, and all results of the last section are thus valid in sectors of non-vanishing topological charge provided we change the coefficient in front of D_1 and let the differential operators act on $(\det\mathcal{M})^{-\nu}\mathcal{Z}_\nu^{\beta=2}$ instead of just $\mathcal{Z}^{\beta=2}$. Thus we find

$$\begin{aligned}&\left[x_i^2\partial_i^2 + (N_f+\nu)x_i\partial_i + \sum_{\substack{i,j\\ i\neq j}}\frac{x_i x_j}{x_i - x_j}(\partial_i - \partial_j)\right] \\ &\times\ (\det\mathcal{M})^{-\nu}\mathcal{Z}_\nu^{\beta=2} = \frac{1}{4}\sum_i x_i(\det\mathcal{M})^{-\nu}\mathcal{Z}_\nu^{\beta=2}. \end{aligned}\tag{A42}$$

Turning to $\beta = 1$ where the measure also changes by $dU \to dU(\det U)^\nu$ we find that the expectation value equation (A18) changes into

$$\begin{aligned}&\left\langle (2N_f - 1+\nu)\mathrm{Tr}\tilde{\mathcal{M}}^\dagger V\right\rangle \\ &= \tfrac{1}{2}\left\langle \mathrm{Tr}\tilde{\mathcal{M}}\tilde{\mathcal{M}}^\dagger - \mathrm{Tr}V\tilde{\mathcal{M}}^\dagger V\tilde{\mathcal{M}}^\dagger\right\rangle. \end{aligned}\tag{A43}$$

In this case we need to replace $\mathcal{Z}^{\beta=1}$ with $(\det\tilde{\mathcal{M}})^{-\nu/2}\mathcal{Z}_\nu^{\beta=1} = (\det\mathcal{M})^{-\nu}\mathcal{Z}_\nu^{\beta=1}$ and we find the differential equation

$$\begin{aligned}&\left[x_i^2\partial_i^2 + (2N_f-1+\nu)x_i\partial_i + 2\textstyle\sum_{\substack{i,j\\ i\neq j}}\frac{x_i x_j}{x_i-x_j}(\partial_i-\partial_j)\right] \\ &\times(\det\mathcal{M})^{-\nu}\mathcal{Z}_\nu^{\beta=1} = \tfrac{1}{4}\textstyle\sum_i x_i(\det\mathcal{M})^{-\nu}\mathcal{Z}_\nu^{\beta=1}. \end{aligned}\tag{A44}$$

Finally, for $\beta = 4$ the change of the measure is $dU \to dU(\det U)^{2\bar\nu}$, with $\bar\nu = N_c\nu$. Hence eq. (A30) becomes

$$\begin{aligned}&\left\langle\left(\tfrac{N_f+1}{2}+\bar\nu\right)\mathrm{Tr}\mathcal{M}^\dagger W\right\rangle \\ &= \tfrac{1}{2}\left\langle \mathrm{Tr}\mathcal{M}\mathcal{M}^\dagger - \mathrm{Tr}W\mathcal{M}^\dagger W\mathcal{M}^\dagger\right\rangle. \end{aligned}\tag{A45}$$

$\mathcal{Z}^{\beta=4}$ should be replaced by $(\det\mathcal{M})^{-\bar\nu}\mathcal{Z}_{\bar\nu}^{\beta=4}$ and thus

$$\begin{aligned}&\left[x_i^2\partial_i^2 + \left(\tfrac{N_f+1}{2}+\bar\nu\right)x_i\partial_i + \tfrac{1}{2}\textstyle\sum_{\substack{i,j\\ i\neq j}}\frac{x_i x_j}{x_i-x_j}(\partial_i-\partial_j)\right] \\ &\times(\det\mathcal{M})^{-\bar\nu}\mathcal{Z}_{\bar\nu}^{\beta=4} = \tfrac{1}{4}\textstyle\sum_i x_i(\det\mathcal{M})^{-\bar\nu}\mathcal{Z}_{\bar\nu}^{\beta=4}. \end{aligned}\tag{A46}$$

APPENDIX B: THE TOPOLOGICAL SUSCEPTIBILITY

In this appendix we calculate the topological susceptibility. This we calculate by allowing a non-vanishing vacuum angle θ and subsequently projecting onto sectors of fixed topological charge by means of a Fourier transform. We consider the classes of symmetry breaking separately.

The case of $\beta = 2$ has been considered previously [17] by means of the same method as we will use here. The result found was

$$\left\langle \nu^2\right\rangle_{\beta=2} = \frac{m_0^2 V F^2}{2N_c}. \tag{B1}$$

Matching of the two finite volume regimes requires that the dependence of the topological susceptibility upon the Dyson index follows the same pattern as the last (non-trivial) part of the diagonal propagator. Thus we expect the topological susceptibility to be twice as large as formula (B1) in the $\beta = 1$ case while being identically the same in the $\beta = 4$ case.

1. $\beta = 1$

A non-vanishing vacuum angle can be absorbed into the mass-matrix $\mathcal{M}$ by letting $\mathcal{M} \to e^{i\theta/N_f}\mathcal{M}$ [21, 22]. We can include the vacuum angle in a shifted Ξ_0. To see this consider how the change of the mass matrix affects the combination

$$\mathrm{ReTr}\mathcal{M}U^\dagger \to \mathrm{ReTr}\left[\mathcal{M}e^{i\theta/N_f}e^{-i\sqrt{2}\xi/F_1}e^{-i\sqrt{2}\varphi_0/F_1}\right], \tag{B2}$$

from which we find that we can include the vacuum angle in a change of variables

$$\varphi_0 \to \varphi_0 - \frac{\theta F_1}{\sqrt{2}N_f}. \tag{B3}$$

Taking the trace of this equation we obtain (note that the dimension of the coset is $2N_f \times 2N_f$)

$$\Xi_0 \to \Xi_0' \equiv \Xi_0 - \sqrt{2}\theta F_1. \tag{B4}$$

Finally we perform the Fourier transform (for large volumes)

$$\begin{aligned}\mathcal{Z}_\nu^{\beta=1} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta e^{-i\nu\theta}\int_{SU(2N_f)/Sp(2N_f)} dU_0 \exp\left[\frac{1}{2}\mathrm{Tr}\mathcal{M}(U_0+U_0^\dagger) - \frac{m_0^2V}{2N_c}\Xi_0'^2\right] \\ &\propto \sqrt{\frac{N_c}{2m_0^2VF_1^2}}e^{-\nu^2N_c/4m_0^2VF_1^2}\int_{U(2N_f)/Sp(2N_f)} dU_0(\det U_0)^{-\nu/2}\exp\left[\frac{1}{2}\mathrm{Tr}\mathcal{M}(U_0+U_0^\dagger)\right],\end{aligned} \tag{B5}$$

from which we identify the topological susceptibility

$$\langle\nu^2\rangle = \frac{2m_0^2VF_1^2}{N_c} = \frac{m_0^2VF^2}{N_c}. \tag{B6}$$

As expected, this is exactly twice the result from the $\beta = 2$ case.

2. $\beta = 4$

The $\beta = 4$ calculation proceeds in the same way. In this case the vacuum angle can be included in the mass matrix by letting $\mathcal{M} \to e^{i\theta/N_cN_f}\mathcal{M}$ [21, 22]. Again this can be included in the field of constant modes, this time by the substitution

$$\varphi_0 \to \varphi_0 - \frac{\theta F_4}{\sqrt{2}N_cN_f}. \tag{B7}$$

Thus for the traced equation we find

$$\Xi_0 \to \Xi_0' \equiv \Xi_0 - \frac{\theta F_4}{\sqrt{2}N_c}. \tag{B8}$$

The projection onto sectors of fixed topological charge is performed as before

$$\begin{aligned}\mathcal{Z}_\nu^{\beta=4} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta e^{-i\nu\theta}\int_{SU(N_f)/SO(N_f)} dU_0 \exp\left[\frac{1}{2}\mathrm{Tr}\mathcal{M}(U_0+U_0^\dagger) - \frac{m_0^2V}{2N_c}\Xi_0'^2\right] \\ &\propto \sqrt{\frac{1}{\langle\bar\nu^2\rangle}}e^{-\bar\nu^2/2\langle\bar\nu^2\rangle}\int_{U(N_f)/SO(N_f)} dU_0(\det U_0)^{-\bar\nu}\exp\left[\frac{1}{2}\mathrm{Tr}\mathcal{M}(U_0+U_0^\dagger)\right],\end{aligned} \tag{B9}$$

with the topological susceptibility given by

$$\langle\bar\nu^2\rangle = \frac{m_0^2VF_4^2}{2N_c} = \frac{m_0^2VF^2}{2N_c}. \tag{B10}$$

Again we define $\bar\nu \equiv N_c\nu$. This result is also as expected.

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